

Coherent and semiclassical states of a free particle

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Abstract

Coherent states (CS) were first introduced and studied in detail for bound motion and discrete-spectrum systems like harmonic oscillators and similar systems with a quadratic Hamiltonian. However, the problem of constructing CS has still not been investigated in detail for the simplest and physically important case of a free particle, for which, besides being physically important, the CS problem is of didactic value in teaching quantum mechanics, with the CS regarded as examples of wave packets representing semiclassical motion. In this paper, we essentially follow the Malkin-Dodonov-Man'ko method to construct the CS of a free nonrelativistic particle. We give a detailed discussion of the properties of the CS obtained, in particular, the completeness relations, the minimization of uncertainty relations, and the evolution of the corresponding probability density. We describe the physical conditions under which free-particle CS can be considered semiclassical states.

Keywords: Coherent states, semiclassical states, free particle.

1 Introduction

Coherent states (CS) play an important role in modern quantum theory as states that provide a natural relation between quantum mechanical and classical descriptions. They have a number of useful properties and, as a consequence a wide range of applications, e.g., in semiclassical description of quantum systems, in quantization theory, in condensed matter physics, in radiation theory, in quantum computations (see, e.g., Refs. [1, 2, 3]). Although there are numerous publications devoted to constructing CS of different systems, a universal definition of a CS and a workable scheme to construct them for an arbitrary physical system is not known. However, we believe that the problem of constructing CS for systems with quadratic Hamiltonians of the general form was completely solved in works by Dodonov and Man'ko, using Malkin and Man'ko integral of motion method, (see [2, 3, 4]). It should be noted that extracting concrete sets of CS and their properties (for a given quadratic system) from their general results one has to perform an additional technical efforts. In this article, we turn our attention to the CS of a free particle. Besides of their physical importance there is a didactic advantage of using free-particle CS in teaching of quantum mechanics, considering them as examples of exact wave packages representing the semiclassical particle motion. In this relation, we note that CS were first introduced and studied in detail for systems with bounded motion and discrete spectrum like harmonic oscillator or a charged particle in a magnetic field. However, for such a simple and physically important system as a free particle, the problem of CS construction was not solved that time. We believe that this situation is explained by the fact that the free particle represents an unbounded motion

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with the continuous energy spectrum, and a generalization of the initial (Glauber) scheme in constructing CS of a harmonic oscillator was not so obvious in this case. Although CS of a free particle, in principle, could be extracted from the above mentioned general results of Dodonov and Man'ko, many authors (ignoring or simply unaware of their results) keep trying to construct CS of a free particle, inventing their own ways. Describing these attempts, we have to cite Refs. [5, 6, 7, 8] devoted to this problem. In our opinion, no single one of these studies completely solves the problem under consideration. Authors of Ref. [6] have quite closely approached the goal, choosing a particular case of initial states for their CS. But even for such initial states, they did not derive an explicit form of time dependent free particle CS and did not study their properties. In fact, their program was realized in the work [5], but the author did not identify his states with some kind of CS. In [7], the authors consider the limit of zero frequency in CS of a harmonic oscillator, deriving a sort of CS for a free particle. Their CS are expressed in terms of sums of Hermite polynomials and the complicated form of the CS hampers their interpretation, study, and applications. Another study [8] treats free particle CS in the framework of a general approach to constructing CS for system with a continuous spectrum. The approach is based on using nonnormalizable fiducial states and involves quite complicated techniques. The authors do not present free-particle well defined for any time instants.

In the present article, in fact following the Dodonov-Man'ko method, we construct different families of generalized CS of a free massive nonrelativistic particle. We discuss in detail properties of the constructed CS, in particular, completeness relations, minimization of uncertainty relations and evolution of the corresponding probability density in time. We describe physical conditions when free particle CS can be considered as semiclassical states.

2 Constructing time-dependent CS of a free particle

2.1 Basic equations

For simplicity, we consider one-dimensional quantum motion of a free nonrelativistic particle of the mass m on the whole real axis $\mathbb{R} = (-\infty, \infty)$. It is described by the Schrödinger equation

$$i\hbar\partial_t\Psi(x,t) = \hat{H}_x\Psi(x,t), \quad x \in \mathbb{R}, \quad (1)$$

where the Hamiltonian \hat{H}_x and the momentum operator \hat{p}_x ,

$$\hat{H}_x = -\frac{\hbar^2}{2m}\partial_x^2 = \frac{\hat{p}_x^2}{2m}, \quad \hat{p}_x = -i\hbar\partial_x, \quad (2)$$

are self-adjoint on their natural domains, [9].

It is useful to introduce the dimensionless variables

$$q = xl^{-1}, \quad \tau = \frac{\hbar}{ml^2}t. \quad (3)$$

Then eq. (1) takes the form

$$\begin{aligned} \hat{S}\psi(q,\tau) &= 0, \quad \hat{S} = i\partial_\tau - \hat{H}, \quad \hat{H}_x = \frac{\hbar^2}{ml^2}\hat{H}, \\ \hat{H} &= \frac{\hat{p}^2}{2}, \quad \hat{p} = -i\partial_q, \quad \psi(q,\tau) = \sqrt{l}\Psi\left(lq, \frac{ml^2}{\hbar}\tau\right), \end{aligned} \quad (4)$$

with $|\Psi(x,t)|^2 dx = |\psi(q,\tau)|^2 dq$. We call the operator \hat{S} the equation operator.

In terms of creation and annihilation operators \hat{a} and \hat{a}^\dagger ,

$$\hat{a} = \frac{\hat{q} + i\hat{p}}{\sqrt{2}}, \quad \hat{a}^\dagger = \frac{\hat{q} - i\hat{p}}{\sqrt{2}}, \quad [\hat{a}, \hat{a}^\dagger] = 1,$$

the Hamiltonian \hat{H} is a quadratic form of these operators

$$\hat{H} = \frac{1}{4} \left[\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger - (\hat{a}^\dagger)^2 - \hat{a}^2 \right]. \quad (5)$$

It cannot be reduced to the first canonical form for a quadratic combination of creation and annihilation operators, which is the oscillator-like form, by any canonical transformation; this indicates that the spectrum of \hat{H} is continuous, see eg. [10].

2.2 Integrals of motion linear in canonical operators \hat{q} and \hat{p}

We construct an integral of motion $\hat{A}(\tau)$ linear in \hat{q} and \hat{p} . The general form of such an integral of motion reads

$$\hat{A}(\tau) = f(\tau) \hat{q} + ig(\tau) \hat{p} + \varphi(\tau), \quad (6)$$

where $f(\tau)$, $g(\tau)$ and $\varphi(\tau)$ are some complex functions of the time τ . For the operator $\hat{A}(\tau)$ to be an integral of motion, it has to commute with the equation operator (4),

$$[\hat{S}, \hat{A}(\tau)] = 0. \quad (7)$$

If the Hamiltonian is self-adjoint, the adjoint operator $\hat{A}^\dagger(\tau)$ is also an integral of motion, $[\hat{S}, \hat{A}^\dagger(\tau)] = 0$.

Substituting representations (6) into eqs. (7), we obtain the following equations for the functions $f(\tau)$, $g(\tau)$, and $\varphi(\tau)$:

$$\dot{f}(\tau) = 0, \quad \dot{g}(\tau) - if(\tau) = 0, \quad \dot{\varphi}(\tau) = 0, \quad (8)$$

where the dot denote derivatives with respect to τ . The general solution of eqs. (8) is

$$f(\tau) = c_1, \quad g(\tau) = c_2 + ic_1\tau, \quad \varphi(\tau) = \text{const}, \quad (9)$$

where c_1 and c_2 are arbitrary constants. Without loss of the generality we can set $\varphi(\tau) = 0$. Thus,

$$\hat{A}(\tau) = c_1 \hat{q} + ig(\tau) \hat{p}, \quad g(\tau) = c_2 + ic_1\tau. \quad (10)$$

The commutator $[\hat{A}(\tau), \hat{A}^\dagger(\tau)]$ reads

$$[\hat{A}(\tau), \hat{A}^\dagger(\tau)] = 2 \text{Re}(g^*(\tau) f(\tau)) = 2 \text{Re}(c_1^* c_2) = \delta. \quad (11)$$

Equations (9) imply that δ is real-valued integral of motion, $\delta = \text{const}$. In what follows we set $\delta = 1$,

$$\delta = 2 \text{Re}(c_1^* c_2) = 1. \quad (12)$$

Let $c_1 = |c_1| e^{i\mu_1}$ and $c_2 = |c_2| e^{i\mu_2}$. Condition (12) then implies that

$$|c_2| |c_1| \cos(\mu_2 - \mu_1) = \frac{1}{2}. \quad (13)$$

Choosing $\delta = 1$, we set $\hat{A}(\tau)$ and $\hat{A}^\dagger(\tau)$ to be annihilation and creation operators,

$$[\hat{A}(\tau), \hat{A}^\dagger(\tau)] = 1. \quad (14)$$

It follows from eqs. (10) and (12) that

$$\begin{aligned} \hat{q} &= g^*(\tau) \hat{A}(\tau) + g(\tau) \hat{A}^\dagger(\tau), \quad g(\tau) = c_2 + ic_1\tau, \\ i\hat{p} &= c_1^* \hat{A}(\tau) - c_1 \hat{A}^\dagger(\tau). \end{aligned} \quad (15)$$

We note that the operators \hat{q} and \hat{p} cannot depend on the constants c_1 , c_2 and time τ . Indeed, using eqs. (6) and (12), one can verify that relations $\partial_\tau \hat{q} = \partial_\tau \hat{p} = \partial_{c_1} \hat{p} = \partial_{c_1} \hat{q} = \partial_{c_2} \hat{p} = \partial_{c_2} \hat{q} = 0$ hold true.

2.3 Time-dependent generalized CS

We consider eigenvectors $|z, \tau\rangle$ of the annihilation operator $\hat{A}(\tau)$ corresponding to the eigenvalue z

$$\hat{A}(\tau) |z, \tau\rangle = z |z, \tau\rangle. \quad (16)$$

In general, z is a complex number. It follows from eqs. (15) and (16) that

$$\begin{aligned} q(\tau) &\equiv \langle z, \tau | \hat{q} | z, \tau \rangle = q_0 + p\tau, \quad q_0 = c_2^* z + c_2 z^*, \\ p(\tau) &\equiv \langle z, \tau | \hat{p} | z, \tau \rangle = i(c_1 z^* - c_1^* z) = p, \\ z &= c_1 q(\tau) + i g(\tau) p = c_1 q_0 + i c_2 p. \end{aligned} \quad (17)$$

Written in q -representation, eqn. (16) becomes

$$[c_1 q + g(\tau) \partial_q] \Phi_z^{c_{1,2}}(q, \tau) = z \Phi_z^{c_{1,2}}(q, \tau), \quad \Phi_z^{c_{1,2}}(q, \tau) \equiv \langle q | z, \tau \rangle. \quad (18)$$

The general solution of this equation has the form

$$\langle q | z, \tau \rangle = \Phi_z^{c_{1,2}}(q, \tau) = \exp \left[-\frac{c_1}{g(\tau)} \frac{q^2}{2} + \frac{zq}{g(\tau)} + \chi(\tau, z) \right], \quad (19)$$

where $\chi(\tau, z)$ is an arbitrary function on τ and z .

We can see that the functions $\Phi_z^{c_{1,2}}(q, \tau)$ can be written in terms of the mean values $q(\tau)$ and $p(\tau)$,

$$\Phi_z^{c_{1,2}}(q, \tau) = \exp \left\{ i p q - \frac{c_1}{2g(\tau)} [q - q(\tau)]^2 + \phi(\tau, z) \right\}. \quad (20)$$

where $\phi(\tau, z)$ is again an arbitrary function on τ and z .

The functions Φ_z satisfy the following equation

$$\hat{S} \Phi_z^{c_{1,2}}(q, \tau) = \lambda(\tau, z) \Phi_z^{c_{1,2}}(q, \tau), \quad (21)$$

where

$$\lambda(\tau, z) = i \dot{\phi}(\tau, z) - \frac{1}{2} \left[p^2 + \frac{c_1}{g(\tau)} \right]. \quad (22)$$

For functions (20) to satisfy Schrödinger equation (4), we have to fix $\phi(\tau, z)$ from the condition $\lambda(\tau, z) = 0$. Thus, for the function $\phi(\tau, z)$, we obtain

$$\phi(\tau, z) = -\frac{i}{2} p^2 \tau - \frac{1}{2} \ln g(\tau) + \ln N, \quad (23)$$

where N is a normalization constant, which we suppose to be real.

The density probability generated by function (20) reads

$$\rho(q, \tau) = |\Phi_z^{c_{1,2}}(q, \tau)|^2 = \frac{N^2}{|g(\tau)|} \exp \left\{ -\frac{[q - q(\tau)]^2}{2 |g(\tau)|^2} \right\}. \quad (24)$$

Considering the normalization integral, we find the constant N ,

$$\int_{-\infty}^{\infty} \rho(q, \tau) dq = 1 \Rightarrow N = (2\pi)^{-1/4}. \quad (25)$$

Thus, normalized solutions of the Schrödinger equation that are eigenfunctions of the annihilation operator $\hat{A}(\tau)$ have the form

$$\Phi_z^{c_{1,2}}(q, \tau) = \frac{1}{\sqrt{\sqrt{2\pi} g(\tau)}} \exp \left\{ i \left(p q - \frac{1}{2} p^2 \tau \right) - \frac{c_1}{g(\tau)} \frac{[q - q(\tau)]^2}{2} \right\} \quad (26)$$

and the corresponding probability density reads

$$\rho_z^{c_{1,2}}(q, \tau) = |\Phi_z^{c_{1,2}}(q, \tau)|^2 = \frac{1}{\sqrt{2\pi}|g(\tau)|} \exp \left\{ -\frac{[q - q(\tau)]^2}{2|g(\tau)|^2} \right\}. \quad (27)$$

In what follows, we call the solutions (26) the time-dependent generalized CS. In fact, we have a family of states parametrized by two complex constants c_1 and c_2 that satisfy restriction (12). As we see in what follows, each family of the generalized CS represent so-called squeezed states. Additional restrictions on the constants c_1 and c_2 transform these states into CS of the free particle (see below).

We note that the generalized CS can be constructed in the Glauber manner, acting by the displacement operator $\mathcal{D}(z, \tau) = \exp \left[z \hat{A}^\dagger(\tau) - z^* \hat{A}(\tau) \right]$ on the vacuum vector $|0, \tau\rangle$ defined as $\hat{A}(\tau)|0, \tau\rangle = 0$:

$$\begin{aligned} \langle q|z, \tau\rangle &= \mathcal{D}(z, \tau) \langle q|0, \tau\rangle = \exp \left[-\frac{|z|^2}{2} \right] \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \langle q|n, \tau\rangle, \\ |n, \tau\rangle &= \frac{[\hat{A}^\dagger(\tau)]^n}{\sqrt{n!}} |0, \tau\rangle, \quad |0, \tau\rangle = \frac{1}{\sqrt{\sqrt{2\pi}g(\tau)}} \exp \left\{ -\frac{c_1}{g(\tau)} \frac{q^2}{2} \right\}. \end{aligned} \quad (28)$$

Functions (28) differ from the set (26) by a constant phase factor only.

Using completeness property of the states $|n, \tau\rangle$,

$$\sum_{n=0}^{\infty} |n, \tau\rangle \langle n, \tau| = 1, \quad \forall \tau, \quad (29)$$

we can find the overlapping and prove the completeness relations for the generalized CS of the free particle

$$\begin{aligned} \langle z', \tau|z, \tau\rangle &= \exp \left(z'^* z - \frac{|z'|^2 + |z|^2}{2} \right), \quad \forall \tau; \\ \int \int \langle q|z, \tau\rangle \langle z, \tau|q'\rangle d^2 z &= \pi \delta(q - q'), \quad d^2 z = d \operatorname{Re} z d \operatorname{Im} z, \quad \forall \tau. \end{aligned} \quad (30)$$

3 Standard deviations, uncertainty relations, and CS of a free particle

Calculating standard deviations $\sigma_q(\tau)$, σ_p , and the quantity $\sigma_{qp}(\tau)$ in the generalized CS, we obtain

$$\begin{aligned} \sigma_q(\tau) &= \sqrt{\langle (\hat{q} - \langle q \rangle)^2 \rangle} = \sqrt{\langle q^2 \rangle - \langle q \rangle^2} = |g(\tau)|, \\ \sigma_p(\tau) &= \sqrt{\langle (\hat{p} - \langle p \rangle)^2 \rangle} = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = |f(\tau)| = |c_1|, \\ \sigma_{qp}(\tau) &= \frac{1}{2} \langle (\hat{q} - \langle q \rangle) (\hat{p} - \langle p \rangle) + (\hat{p} - \langle p \rangle) (\hat{q} - \langle q \rangle) \rangle \\ &= i [1/2 - g(\tau) f^*(\tau)]. \end{aligned} \quad (31)$$

It is easy to see that the generalized CS minimize the Robertson-Schrödinger uncertainty relation [11],

$$\sigma_q^2(\tau) \sigma_p^2 - \sigma_{qp}^2(\tau) = \frac{1}{4}. \quad (32)$$

This means that these states are squeezed states for any time instant.

We evaluate the Heisenberg uncertainty relation in the generalized CS taking constraint (12) into account:

$$\sigma_q(\tau) \sigma_p(\tau)|_{2\text{Re}(c_1^* c_2)} = \sqrt{\frac{1}{4} + \left[|c_2| |c_1| \sin(\mu_2 - \mu_1) + |c_1|^2 \tau\right]^2} \geq \frac{1}{2}. \quad (33)$$

Using (31), we then find $\sigma_q(0) = \sigma_q = |c_2|$ and $\sigma_p(0) = \sigma_p = |c_1|$, and hence at $\tau = 0$ this relation become

$$\sigma_q \sigma_p|_{2\text{Re}(c_1^* c_2)} = |c_2| |c_1| = \sqrt{\frac{1}{4} + [|c_2| |c_1| \sin(\mu_2 - \mu_1)]^2}. \quad (34)$$

We see that $|c_i| \neq 0$, $i = 1, 2$ and the left-hand side of (34) is minimal if $\mu_1 = \mu_2 = \mu$, which provides the minimization of the Heisenberg uncertainty relation in the generalized CS at the initial time instant,

$$\sigma_q \sigma_p = \frac{1}{2}. \quad (35)$$

In what follows, we consider the free-particle generalized CS with the restriction $\mu_1 = \mu_2$. Such states are simply called CS of a free particle.

Now, constraint (12) take the form

$$|c_2| |c_1| = 1/2 \implies c_2^* = c_1^{-1}/2. \quad (36)$$

We see that the constant μ don't enter the CS (26). Thus, we set $\mu = 0$ in what follows. Then

$$\begin{aligned} c_2 = |c_2| = \sigma_q, \quad c_1 = |c_1| = \sigma_p = 1/(2\sigma_q), \\ g(\tau) = \left(\sigma_q + \frac{i\tau}{2\sigma_q}\right), \quad \sigma_q(\tau) = |g(\tau)| = \sqrt{\sigma_q^2 + \frac{\tau^2}{4\sigma_q^2}}. \end{aligned} \quad (37)$$

From eqn. (37), we conclude that for any τ , the Heisenberg uncertainty relation in the CS takes the form

$$\sigma_q(\tau) \sigma_p = \frac{1}{2} \sqrt{1 + \frac{\tau^2}{4\sigma_q^4}} \geq \frac{1}{2} \quad (38)$$

and CS of a free particle read

$$\Phi_z^{\sigma_q}(q, \tau) = \frac{\exp\left\{i\left(pq - \frac{p^2}{2}\tau\right) - \frac{[q - q(\tau)]^2}{4(\sigma_q^2 + i\tau/2)}\right\}}{\sqrt{\left(\sigma_q + \frac{i\tau}{2\sigma_q}\right)} \sqrt{2\pi}}. \quad (39)$$

In fact, we have a family of CS parametrized by one real parameter σ_q . Each set of CS in the family has its specific initial standard deviations $\sigma_q > 0$. CS from a family with a given σ_q are labeled by quantum numbers z ,

$$z = \frac{q_0}{2\sigma_q} + i\sigma_q p, \quad (40)$$

which are in one to one correspondence with trajectory initial data q_0 and p ,

$$q_0 = 2\sigma_q \text{Re } z, \quad p = \frac{\text{Im } z}{\sigma_q}. \quad (41)$$

The probability densities that corresponds to the CS are

$$\rho_z^{\sigma_q}(q, \tau) = \frac{1}{\sqrt{\left(\sigma_q^2 + \frac{\tau^2}{4\sigma_q^2}\right)} 2\pi} \exp \left\{ -\frac{1}{2} \frac{[q - q(\tau)]^2}{\sigma_q^2 + \frac{\tau^2}{4\sigma_q^2}} \right\}. \quad (42)$$

It can be seen that at any time instant τ , probability densities (42) are given by Gaussian distributions with standard deviations $\sigma_q(\tau)$. The mean values $\langle q \rangle = q(\tau) = q_0 + p\tau$ are moving along the classical trajectory with the particle velocity p . The maxima of the probability densities move with the same velocity (42).

Figure 1 plots function (42) with $\sigma = 2^{-1/2}$, $p = 2$, $q_0 = 0$, for two time instants, $\tau = 0$ and $\tau = 1$,

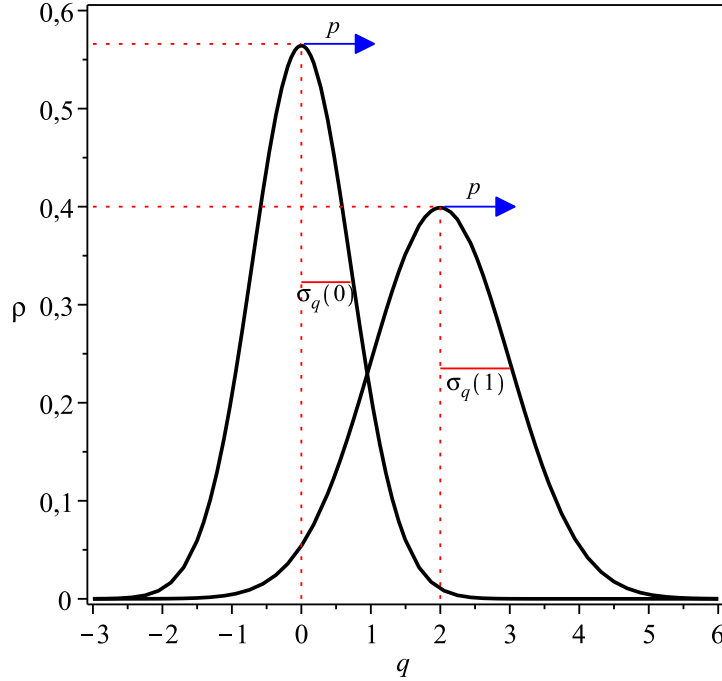


Figure 1: Evolution of the probability densities.

Let us compare the CS (39) with the plane waves

$$\Phi_p(q, \tau) = \frac{1}{\sqrt{2\pi}} \exp \left[i \left(pq - \frac{p^2}{2} \tau \right) \right]. \quad (43)$$

Both sets of functions are solutions of the Schrödinger equation for the free particle. The CS do belong to $L^2(\mathbb{R})$, whereas the plane waves do not. A plane wave propagates with the phase velocity that is $p/2$. The CS set generates the probability density that propagates exactly with the particle velocity p . We can say that CS (39) represent wave packets that allow establishing a natural connection between the classical and quantum description of free particles. Depending of parameters of the CS, some of them can be treated as semiclassical states of free particles, some cannot, because they describe pure quantum states (see bellow).

4 Semiclassical CS of free particle

To consider the question which CS can be treated as representing a semiclassical particle motion, we have to return to the initial dimensional variables x and t (3) and to the initial wave function $\Psi(x, t)$ written in these variables (4). Taking into account that

$$\begin{aligned} x(t) &= lq(\tau) = x_0 + \frac{p_x}{m}t, \quad p = \frac{l}{\hbar}p_x, \\ \sigma_x(0) &= l\sigma_q(0) = l\sigma = \sigma_x, \quad \sigma_x^2(t) = \sigma_x^2 + \frac{\hbar^2}{4m^2\sigma_x^2}t^2, \end{aligned} \quad (44)$$

we obtain

$$\begin{aligned} \Psi(x, t) &= \frac{1}{\sqrt{\left(\sigma_x + \frac{i\hbar}{2m\sigma_x}t\right)}\sqrt{2\pi}} \exp \left\{ \frac{i}{\hbar} \left(p_x x - \frac{p_x^2}{2m}t \right) - \frac{[x - x(t)]^2}{4\left(\sigma_x^2 + \frac{\hbar^2}{4m^2\sigma_x^2}t^2\right)} \right\}, \\ \rho(x, t) &= |\Psi(x, t)|^2 = \frac{1}{\sqrt{\left(\sigma_x^2 + \frac{\hbar^2}{4m^2\sigma_x^2}t^2\right)}2\pi} \exp \left\{ -\frac{1}{2} \frac{[x - x(t)]^2}{\sigma_x^2 + \frac{\hbar^2}{4m^2\sigma_x^2}t^2} \right\}. \end{aligned} \quad (45)$$

Semiclassical motion implies that the form of the distribution (45) changes slowly with time t in a certain sense. This form changes due to the change in the quantity $\frac{\hbar^2}{4m^2\sigma_x^2}t^2$ with time, which is responsible for the change of $\sigma_x^2(t)$ (see eq. (44)). We suppose that in case of semiclassical motion, this quantity is much less than the square of the distance that the particle travel in the same time. We then have the inequality

$$\frac{\hbar^2}{4m^2\sigma_x^2}t^2 \ll \left(\frac{p_x}{m}t\right)^2 \implies p_x \gg \frac{\hbar}{2\sigma_x} \sim v \gg \frac{\hbar}{2m\sigma_x}, \quad (46)$$

which can be rewritten in another form:

$$\lambda \ll 4\pi\sigma_x, \quad \lambda = \frac{2\pi\hbar}{p_x}, \quad (47)$$

where λ is the Compton wavelength of the particle. Hence, the CS of a free particle can be considered semiclassical states if the Compton wavelength of the particle is much less than the coordinate standard deviation σ_x at the initial instant. It is known that in a cyclotron, nonrelativistic electrons are moving with velocities $v \simeq 10^3 \frac{\text{m}}{\text{s}}$. Then, according to eq. (46), CS of such electrons with $2\sigma_x \simeq 10^{-7}\text{m}$ can be treated as semiclassical states.

It should be noted that similar criteria of the semiclassicality were used in theory of potential scattering [12] and for classifying CS in a magnetic-solenoid field [13].

5 Some concluding remarks

In this article, we have studied different types of generalized CS of a free massive nonrelativistic particle and established properties of these states such as the completeness relations, the minimization of uncertainty relations, and the evolution of the corresponding probability densities in time. Among all these types of generalized CS, families of states are naturally distinguished which we suggest identifying with the CS of a free massive nonrelativistic particle. These CS families are parameterized by one real-valued parameter, the coordinate standard deviation σ_q at the initial time instant. The CS from a family with a given σ_q form a complete system of functions and are labeled by a complex-valued quantum number z , which is in a one-to-one correspondence with the initial data of the corresponding trajectory of the coordinate mean value. CS minimize the Robertson-Schrodinger uncertainty

relation at all time instants and the Heisenberg uncertainty relations at the initial instant. The smaller the coordinate standard deviation $\sigma_q(\tau)$ at the initial instant is, the faster τ grows with time at an arbitrary instant. At any time instant τ , the probability density corresponding to free-particle CS is given by Gaussian distributions with standard deviations $\sigma_q(\tau)$. The coordinate mean value propagates along the classical trajectory with the mean particle velocity. The probability density maximum propagates with the same velocity. The constructed CS are wave packets that are solutions of the Schrödinger equation for a free particle. They belong to the Hilbert space $L^2(\mathbb{R})$, whereas plain waves do not belong to this space. The CS allow establishing a natural relation between the classical and quantum descriptions of free particles. Depending on the parameters of the CS, some of them can be considered semiclassical states of free particles, and some of them cannot, inasmuch as the latter are purely quantum states. We provide arguments in favor of the fact that free-particle CS can be considered semiclassical states when the Compton wavelength is much less than the standard coordinate deviation at the initial instant. The suggested CS can be apparently identified with the asymptotic free states in nonrelativistic quantum scattering theory.

We believe it is useful for a lecture course in quantum mechanics to complete the description of quantum motion of a free particle with the free-particle CS as an example of exact wave packets, which, under certain conditions, admit a semiclassical description of such a particle, and which allow illustrating a large number of general principles of quantum mechanics, such as the minimization of uncertainty relations. The acquaintance of an audience with free-particle CS would naturally make it easier to understand the CS of an oscillator and other quantum systems.

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